

Notes on partial conjugation

Chuying Fang and Xuhua He

To Fiona with adoration

ABSTRACT. In this notes, we will give an exposition of some results on the method of partial conjugation action. We first discuss the partial conjugation action of a parabolic subgroup of a Coxeter group. We then discuss some applications to Lusztig's G -stable pieces and its affine generalization. We also discuss some recent work on the σ -conjugacy classes of loop groups and affine Deligne-Lusztig varieties.

Introduction

Let G be a connected reductive algebraic group over an algebraically closed field \mathbb{k} . Let X be a “nice” $G \times G$ -variety (e.g., a wonderful compactification of an adjoint group). In [L1], Lusztig introduced a partition of X into finitely many locally closed subvarieties, which are stable under the diagonal G -action. These subvarieties are called G -stable pieces. Each G -stable piece is a union of orbits for the diagonal action of G and there is a natural bijection between these G -orbits and twisted conjugacy classes of a Levi factor of G .

The partition of X into G -stable pieces is central to Lusztig's work on parabolic character sheaves and the work of Evens and Lu [EL] on Poisson geometry. More recently, Pink, Wedhorn and Ziegler [PWZ] used G -stable pieces to study some problems in arithmetic algebraic geometry.

The notion of G -stable pieces is relative new and a little hard to penetrate as the statements are fairly technical. The main purpose of this article is to explain how the method of partial conjugation action introduced by the second named author in [H3] can be used to understand Lusztig's G -stable pieces. We will also discuss some recent progress on the study of loop groups and affine Deligne-Lusztig varieties using the method of partial conjugation action.

1. Partial conjugation in a Coxeter group

1.1. We first recall the definition of Coxeter groups.

Let S be a finite set and $(m_{ij})_{i,j \in S}$ be a matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$ for all $i \neq j$. Let W be a group defined by the

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generators s_i for $i \in I$ and the relations $(s_i s_j)^{m_{ij}} = 1$ for $i, j \in I$ with $m_{ij} < \infty$. We say that (W, S) is a *Coxeter group*. Sometimes we just call W itself a Coxeter group.

We denote by ℓ the length function and \leq the Bruhat order. For $J \subset S$, we denote by W_J the standard parabolic subgroup of W generated by J and by W^J (resp. ${}^J W$) the set of minimal coset representatives in W/W_J (resp. $W_J \backslash W$). For $J, K \subset I$, we simply write $W^J \cap {}^K W$ as ${}^K W^J$.

1.2. Let $J, J' \subset S$ and $\delta : W_J \rightarrow W_{J'}$ such that $\delta(J) = J'$. Now we consider the δ -twisted partial conjugation of W_J on W defined by

$$w \cdot_{\delta} w' = \delta(w) w' w^{-1}.$$

If $J = J'$ and δ is the identity map, then we simply write \cdot for \cdot_{δ} .

For any subset V of W , we set

$$W_J \cdot_{\delta} V = \{\delta(w) w' w^{-1}; w \in W_J, w' \in V\}.$$

The classification of W_J -orbits on W for this twisted partial conjugation action is as follows. See [H3, Section 2].

THEOREM 1.1. *For any $w \in W^J$, set*

$$I(J, w, \delta) = \max\{K \subset J; \forall k \in K, \exists k' \in \delta(K), w s_k = s_{k'} w\}.$$

Then

$$(1) W = \sqcup_{w \in W^J} W_J \cdot_{\delta} (w W_{I(J, w, \delta)}).$$

(2) *The inclusion $W_{I(J, w, \delta)} \rightarrow W_J \cdot_{\delta} (w W_{I(J, w, \delta)})$ induces a natural bijection between the $\text{Ad}(w)^{-1} \circ \delta$ -twisted conjugacy classes on $W_{I(J, w, \delta)}$ and the W_J -orbits on $W_J \cdot_{\delta} (w W_{I(J, w, \delta)})$.*

We'll say a few words on the $\text{Ad}(w)^{-1} \circ \delta$ -twisted conjugation action in (2) of the above theorem. It is easy to see from the definition of $I(J, w, \delta)$ that for any $v \in W_{I(J, w, \delta)}$, $w^{-1} \delta(v) w \in W_{I(J, w, \delta)}$. The $\text{Ad}(w)^{-1} \circ \delta$ -twisted conjugation action of $W_{I(J, w, \delta)}$ on itself is defined to be $v \cdot_{\text{Ad}(w)^{-1} \circ \delta} v' = w^{-1} \delta(v) w v' w^{-1} \in W_{I(J, w, \delta)}$. Moreover, for any $v, v' \in W_{I(J, w, \delta)}$,

$$w(v \cdot_{\text{Ad}(w)^{-1} \circ \delta} v') = w(w^{-1} \delta(v) w v' w^{-1}) = \delta(v) w v' w^{-1} = v \cdot_{\delta} (w v').$$

1.3. In particular, we have a map

$$\pi_{J, \delta} : W \rightarrow W^J, W_J \cdot_{\delta} (w W_{I(J, w, \delta)}) \mapsto w$$

that is constant on each orbit for the δ -twisted partial conjugation action of W_J .

Below are two examples. Here red color for the elements in W^J and blue color for the elements in W_J .

EXAMPLE 1.2. $W = S_4$, $J = J' = \{1, 2\}$ and δ is the identity map.

The image of $w = s_2 s_3 s_2 s_1 s_2$ is obtained as follows.

$$\begin{aligned} \textcolor{red}{s_2} \textcolor{red}{s_3} \textcolor{red}{s_2} \textcolor{red}{s_1} \textcolor{red}{s_2} &\xrightarrow{\text{conj. by } s_2} \textcolor{red}{s_3} \textcolor{red}{s_2} \textcolor{red}{s_1} \xrightarrow{\text{conj. by } s_1} \textcolor{red}{s_3} \textcolor{red}{s_1} \textcolor{red}{s_2} \xrightarrow{\text{conj. by } s_2} \textcolor{red}{s_2} \textcolor{red}{s_3} \textcolor{red}{s_1} \\ &\xrightarrow{\text{conj. by } s_1} \textcolor{red}{s_1} \textcolor{red}{s_2} \textcolor{red}{s_3} \in W^J. \end{aligned}$$

So $\pi_{J, \delta}(s_2 s_3 s_2 s_1 s_2) = s_1 s_2 s_3$.

EXAMPLE 1.3. $W = S_4$, $J = J' = \{1, 3\}$ and δ is the identity map. The image of $w = s_2 s_1 s_3 s_2 s_1$ is obtained as follows.

$$\textcolor{red}{s_2 s_1 s_3 s_2 s_1} \xrightarrow{\text{conj. by } s_1} \textcolor{red}{s_2 s_1 s_3 s_2 s_3} \xrightarrow{\text{conj. by } s_3} \textcolor{red}{s_2 s_1 s_3 s_2 s_1}.$$

In this case, $I(J, s_2 s_1 s_3 s_2, \delta) = \{1, 3\}$ and $\pi_{J, \delta}(s_2 s_1 s_3 s_2 s_1) = s_2 s_1 s_3 s_2$.

1.4. In order to use the partial conjugation of Weyl group to understand structure of reductive groups, we need additional properties related to the partial conjugation.

The following notion is motivated by [GP].

Let $w, w' \in W$. We write $w \xrightarrow{i} w'$ for $i \in J$ if $w' = s_{\delta(i)} w s_i$ and $\ell(w') \leq \ell(w)$.

We write $w \rightarrow_{J, \delta} w'$ if there is a sequence $w = w_1 \xrightarrow{i_1} w_2 \xrightarrow{i_2} \cdots \xrightarrow{i_n} w_{n+1} = w'$, where $i_1, \dots, i_n \in J$. We write $w \approx_{J, \delta} w'$ if $w \rightarrow_{J, \delta} w'$ and $w' \rightarrow_{J, \delta} w$. It is easy to see that $w \approx_{J, \delta} w'$ if and only if $w \rightarrow_{J, \delta} w'$ and $\ell(w) = \ell(w')$. If $J = S$ and δ is the identity map, then we simply write \rightarrow for $\rightarrow_{J, \delta}$ and \approx for $\approx_{J, \delta}$.

We say that $w \in W$ is terminal with respect to (J, δ) if for any $w' \in W$ with $w \rightarrow_{J, \delta} w'$, we have that $w \approx_{J, \delta} w'$.

Now we have the following results [H3, Section 3].

THEOREM 1.4. *Let $w \in W$ with $\pi_{J, \delta} = w'$. Then there exists $x \in W_{I(J, w, \delta)}$ such that $w \rightarrow_{J, \delta} w' x$. If moreover, $\ell(w) = \ell(w')$, then $w \approx_{J, \delta} w'$.*

In fact, if W_J is a finite Coxeter group, then we can choose x in such a way that $w' x$ is a minimal length element in the δ -twisted conjugacy class of W_J that contains w .

1.5. By [H3, Corollary 4.5], for any δ -twisted W_J -conjugacy class \mathcal{O} that contains an element in W^J , the following conditions are equivalent:

- (1) v is a minimal element in \mathcal{O} with respect to the restriction to \mathcal{O} of the Bruhat order on W .
- (2) v is an element of minimal length in \mathcal{O} .

We denote by \mathcal{O}_{\min} the set of elements in \mathcal{O} that satisfy the above conditions. As in [H3, 4.7], we have a natural partial order $\leq_{J, \delta}$ on W^J defined as follows:

Let $w, w' \in W^J$. Then $w \leq_{J, \delta} w'$ if for some (or equivalently, any) $v' \in (W_J \cdot \delta w')_{\min}$, there exists $v \in (W_J \cdot \delta w)_{\min}$ such that $v \leq v'$.

In general, for $w \in W^J$ and $w' \in W$, we write $w \leq_{J, \delta} w'$ if there exists $v \in (W_J \cdot \delta w)_{\min}$ such that $v \leq w'$.

2. Partial conjugation in groups with (B, N) -pair

2.1. We first recall the definition of a (B, N) -pair.

A group G has a (B, N) -pair if the following axioms hold:

- (1) G is generated by the subgroups B and N .
- (2) $B \cap N$ is a normal subgroup of N .
- (3) The group $W = N/B \cap N$ is generated by a set of elements s_i of order 2, for i in some nonempty set S .
- (4) For any $i \in S$ and $w \in W$, $s_i B w \subset B s_i w B \cup B w B$.
- (5) $s_i \notin N_G(B)$ for all $i \in S$.

In this case, (W, S) is a Coxeter group and $G = \sqcup_{w \in W} BwB$. Moreover, for any $i \in S$ and $w \in W$, we have that

$$Bs_iBwB = \begin{cases} Bs_iwB, & \text{if } s_iw > w; \\ Bs_iwB \sqcup BwB, & \text{if } s_iw < w. \end{cases}$$

$$BwBs_iB = \begin{cases} Bws_iB, & \text{if } ws_i > w; \\ Bws_iB \sqcup BwB, & \text{if } ws_i < w. \end{cases}$$

2.2. Let $J, J' \subset S$ and $L_J, L_{J'}$ the corresponding Levi subgroups of G . Let $\sigma : L_J \rightarrow L_{J'}$ a isomorphism of (abstract) groups that sends $B \cap L_J$ to $B \cap L_{J'}$ and $H \cap L_J$ to $H \cap L_{J'}$. Then σ induces an isomorphism $\delta : W_J \rightarrow W_{J'}$ such that $\delta(J) = J'$. Define the σ -twisted partial conjugation action of L_J on G by $l \cdot_\sigma g = \sigma(l)gl^{-1}$.

The following Lemma is crucial in the study of σ -twisted partial conjugation action.

LEMMA 2.1. *We keep the notations as above. Let $w, w' \in W$.*

(1) *If $w \approx_{J, \delta} w'$, then $L_J \cdot_\sigma BwB = L_J \cdot_\sigma Bw'B$.*

(2) *If w is not terminal with respect to (J, δ) , then*

$$L_J \cdot_\sigma BwB \subset \cup_{v \in W, \ell(v) < \ell(w)} L_J \cdot_\sigma BvB.$$

Now similar to the decomposition in Theorem 1.1, we have the following result. The proof is based on combinatorial properties of partial conjugation in Weyl group.

THEOREM 2.2. $G = \cup_{w \in W^J, x \in W_{I(J, w, \delta)}} L_J \cdot_\sigma BwxB$.

Proof. Notice that $G = \sqcup_{v \in W} BvB$. Thus it suffices to prove that for any v ,

$$(1) \quad BvB \subset \cup_{w \in W^J, x \in W_{I(J, w, \delta)}} L_J \cdot_\sigma BwxB.$$

We argue by induction on $\ell(v)$. Let $w_1 = \pi_{J, \delta}(v)$. Then by Theorem 1.4, there exists $x_1 \in W_{I(J, w_1, \delta)}$ such that $v \rightarrow_{J, \delta} w_1 x_1$. Now by Lemma 2.1,

$$(2) \quad BvB \subset L_J \cdot_\sigma Bw_1 x_1 B \cup \cup_{v' \in W, \ell(v') < \ell(v)} L_J \cdot_\sigma Bv'B.$$

By induction hypothesis, for any v' with $\ell(v') < \ell(v)$,

$$(3) \quad Bv'B \subset \cup_{w \in W^J, x \in W_{I(J, w, \delta)}} L_J \cdot_\sigma BwxB.$$

Now (1) follows from (2) and (3). This finishes the proof. \square

It is very interesting to consider when J is of finite type. We discuss in the following two sections the case where G is of finite type and affine type. These results can also be generalized to arbitrary Kac-Moody groups. We do not go into this in the article.

3. Partial conjugation: finite type

The notion of G -stable pieces was introduced by Lusztig in [L1] in the study of parabolic character sheaves. A simpler formulation was given in [H1] and the closure relation was found in [H2]. Different approaches were obtained by Evens and Lu in [EL] and by Springer in [Sp3]. The notion was later generalized by Lu and Yakimov to R -stable pieces in [LY]. We refer to the survey article of Springer [Sp1] for some applications of G -stable pieces in representation theory.

3.1. In this section, assume that G is a reductive group over an algebraically closed field k . Let B be a Borel subgroup of G and $T \subset B$ a maximal torus. We denote by W the Weyl group of G and S the set of simple roots. For any subset J of W , we denote by w_0^J the maximal element in W_J .

Assume that $J \subset S$ and $\sigma : L_J \rightarrow L_{J'}$ is an isomorphism of algebraic groups. Then for any $w \in W^J$, the map $L_{I(J,w,\delta)} \rightarrow L_{I(J,w,\delta)}$, $l \mapsto w^{-1}\sigma(l)w$ is again an isomorphism of algebraic groups. Therefore the map

$$L_{I(J,w,\delta)} \times (B \cap L_{I(J,w,\delta)}) \rightarrow L_{I(J,w,\delta)}, \quad (l, b) \mapsto w^{-1}\sigma(l)wbl^{-1}$$

is surjective (see [St]). As a consequence, one can see that

$$L_{I(J,w,\delta)} \cdot_\sigma BwxB \subset L_{I(J,w,\delta)}BwB = L_{I(J,w,\delta)} \cdot_\sigma BwB$$

for all $x \in W_{I(J,w,\delta)}$. Now by Theorem 2.2, $G = \cup_{w \in W^J} L_J \cdot_\sigma BwB$. In fact, a detailed analysis shows that this union is a disjoint union (see [L1], [Sp3]).

THEOREM 3.1. $G = \sqcup_{w \in W^J} L_J \cdot_\sigma BwB$.

Now we describe the closure relations between these subvarieties of G . The proof is based on combinatorial property of partial conjugation in Weyl group. Details can be found in [H3, Section 5].

THEOREM 3.2. For any $w \in W$,

$$\overline{L_J \cdot_\sigma BwB} = \sqcup_{w' \in W^J, w' \leq_{J,\delta} w} L_J \cdot_\sigma Bw'B.$$

3.2. We set $R_{J,\sigma} = \{(lu, \sigma(l)u'); l \in L_J, u \in U_{P_J}, u' \in U_{P_{\delta(J)}}\} \subset G \times G$, where U_{P_J} is the unipotent radical of the standard parabolic subgroup P_J and $U_{P_{\delta(J)}}$ is the unipotent radical of $P_{\delta(J)}$. The action of $R_{J,\delta}$ on G is defined by $(lu, \sigma(l)u', g) \mapsto lug(\sigma(l)u')^{-1}$. Then it is easy to see that for any $w \in W$, $R_{J,\sigma} \cdot BwB = (L_J \cdot_\delta BwB)^{-1}$. Thus

- (1) $G = \sqcup_{w \in W^J} R_{J,\sigma} \cdot Bw^{-1}B$.
- (2) For any $w \in W^J$, $\overline{R_{J,\sigma} \cdot BwB} = \sqcup_{w' \in W^J, w' \leq_{J,\delta} w} R_{J,\sigma} \cdot B(w')^{-1}B$.

3.3. Let G_Δ be the diagonal image of G in $G \times G$. Then we may regard G as $G \times G/G_\Delta$ via $g \mapsto (1, g)G_\Delta/G_\Delta$ and thus there exists a natural bijection of the $R_{J,\sigma}$ -orbits on $G \cong (G \times G)/G_\Delta$ and G_Δ -orbits on $(G \times G)/R_{J,\sigma}$. This bijection is good in the sense of [Sp3, Lemma 1.6]. Under this bijection, the subset $R_{J,\delta} \cdot Bw^{-1}B$ of G corresponds to the subset $G_\Delta \cdot (Bw, B)R_{J,\sigma}/R_{J,\sigma}$ of $(G \times G)/R_{J,\sigma}$. For any $w \in W^J$, we write $Z_{J,w,\sigma}$ for $G_\Delta \cdot (Bw, B)R_{J,\sigma}/R_{J,\sigma}$ and call it a G -stable piece of $(G \times G)/R_{J,\sigma}$.

Below are some properties of the G -stable pieces.

- (1) $(G \times G)/R_{J,\sigma} = \sqcup_{w \in W^J} Z_{J,w,\sigma}$. See [L1].
- (2) For any $w \in W^J$, $Z_{J,w,\sigma}$ is a locally closed smooth subvariety of $(G \times G)/R_{J,\sigma}$ of codimension $\ell(w_0^J w_0) - \ell(w)$. See [L1].
- (3) There is a natural bijection between the G_Δ -orbits on $Z_{J,w,\sigma}$ and the $\sigma \circ \text{Ad}(w)$ -twisted conjugacy classes of $L_{I(J,w,\delta)}$. See [L1].
- (4) For any $w \in W^J$, $\overline{Z_{J,w,\sigma}} = \sqcup_{w' \in W^J, w' \leq_{J,\delta} w} Z_{J,w',\sigma}$. See [H3].
- (5) For any $w \in W^J$, $\overline{Z_{J,w,\sigma}}$ is a semi-normal variety. See [HT].

4. Some variations

4.1. We keep the notations in Section 3.

The action of $-w_0$ on the set of simple roots gives an involution $*$: $S \rightarrow S$. Let $J \subset S$ and P_J^- be the parabolic subgroup of G opposite to the standard parabolic P_J . Then $P_J^- = {}^{w_0^J}w_0 P_{J^*}$. Define

$$Z_J = (G \times G) \times_{P_J^- \times P_J} L_J,$$

here the action of $P_J^- \times P_J$ on $G \times G \times L_J$ is defined by $(p, p') \cdot (g, g', l) = (gp^{-1}, g'(p')^{-1}, \pi_{P_J^-}(p)l\pi_{P_J}(p'))$, where $\pi_{P_J^-} : P_J^- \rightarrow L_J$ and $\pi_{P_J} : P_J \rightarrow L_J$ are projection maps. It is easy to see that $Z_J \cong (G \times G)/R_J^-$, where $R_J^- = \{(lu, lu'); l \in L_J, u \in U_{P_J^-}, u' \in U_{P_J}\}$.

Now define $\sigma : L_{J^*} \rightarrow L_J$ by $\sigma(l) = (w_0^J w_0)l(w_0^J w_0)^{-1}$. Then

$$R^- = (w_0^J w_0, 1)R_{J^*, \sigma}(w_0^J w_0, 1)^{-1}.$$

Hence the isomorphism $G \times G \rightarrow G \times G$, $(g, g') \mapsto (g(w_0^J w_0)^{-1}, g')$ induces an isomorphism $G \times G/R_{J^*, \sigma} \rightarrow Z_J$.

Now for any $w \in W^J$, we write $Z_{J,w}$ for $G_\Delta \cdot (Bw, B)R_J^-/R_J^-$ and call it a G -stable piece of Z_J .

4.2. In this subsection, we assume that G is of adjoint type. In the same manner as Z_J , we define $X_J = (G \times G)_{P_J^- \times P_J} L_J/Z(L_J)$. This is a boundary $G \times G$ -orbit of the wonderful compactification X of G , see [CP].

We denote by h_J the image of $(1, 1, 1)$ in X_J . For $w \in W^J$, set

$$X_{J,w} = G_\Delta \cdot (Bw, B)h_J$$

and call it a G -stable piece of X . Below are some properties of $X_{J,w}$.

- (1) $X = \sqcup_{J \subset S, w \in W^J} X_{J,w}$. See [L1].
- (2) For any $J \subset S, w \in W^J$, $X_{J,w}$ is a locally closed smooth subvariety of X of codimension $\ell(w) + \sharp(S - J)$. See [L1].
- (3) For any $J \subset S, w \in W^J$, $\overline{X_{J,w}} = \sqcup_{K \subset J, w' \in W^K, w \leq_{J, id} w'} X_{K,w'}$. See [H2].
- (4) For any $J \subset S, w \in W^J$, $\overline{X_{J,w}}$ is a semi-normal variety. See [HT].
- (5) If $\overline{X_{J,w}}$ contains finitely many G_Δ -orbits, then it admits a cellular decomposition. See [H2].

4.3. In the rest of this section, we assume that the characteristic of the ground field \mathbb{k} is positive. Instead of considering isomorphism of algebraic groups $\sigma : L_J \rightarrow L_{J'}$ in §3.1, we may consider the morphism $\sigma \circ F : L_J \rightarrow L_{J'}$, where $F : L_J \rightarrow L_J$ is a Frobenius morphism. Now using Lang's theorem, one can show that for any $w \in W^J$, $R_{J, \sigma \circ F}$ acts transitively on $R_{J, \sigma \circ F} \cdot Bw^{-1}B$.

Similarly, for any Frobenius morphism $F : G \rightarrow G$ and any $w \in W^J$, G_F acts transitively on $G_F \cdot (Bw, B)R_{J, \sigma} \subset G \times G/R_{J, \sigma}$, where $G_F = \{(g, F(g)); g \in G\} \subset G \times G$. The closure relation between the G_F -orbits is still given by $\leq_{J, \delta}$. This is used in the study of finer Deligne-Lusztig varieties in a partial flag variety [H4]. This is also used by Pink, Wedhorn and Ziegler, in the study of F -zips [PWZ].

4.4. Now we assume that $J = J_1 \sqcup J_2$ and $W_J = W_{J_1} \times W_{J_2}$. We consider the actions of $W_{J_2} \times W_{\delta(J_2)}$ on the set of W_{J_1} -conjugacy classes of W and its applications to reductive groups. We only list the results below. The details will appear in a future work.

For $w \in W^{J_1}$, define $K_i(w)$ (for $i \geq 0$) as follows: $K_0(w) = J_2$ and $K_i(w) = wK_{i-1}(w) \cap J_1$ for $i \geq 1$. Set

$$\mathcal{W}(J_1, J_2) = \{w \in W^{J_1}; w \in K_i(w)W \text{ for all } i\}.$$

This set first appeared in the study of conjugacy classes in reductive monoid. See [Pu].

We have that

$$(1) \quad W = \sqcup_{w \in \mathcal{W}(J_1, J_2)} W_{J_2} \pi_{J_1, id}^{-1}(w) W_{J_2}.$$

(2) For any $w \in \mathcal{W}(J_1, J_2, \delta)$, the $(W_{\delta(J_2)} \times W_{J_2})(W_{J_1})_{id}$ -orbits on $W_{J_2} \pi_{J_1, id}^{-1}(w) W_{J_2}$ are in bijection with the $(W_{J_1})_{id}$ -orbits on $\pi_{J_1, \delta}^{-1}(w)$, i.e. $\text{Ad}(w)$ -twisted conjugacy classes on $W_{I(J_1, w, id)}$.

Now let $R_{J_1, J_2} = \{(lp_1, lp_2; l \in L_{J_1}, p_1, p_2 \in U_{P_{J_1 \cup J_2}} L_{J_2})\}$. Then

$$(G \times G)/R_{J_1, J_2} = \sqcup_{w \in \mathcal{W}(J_1, J_2)} G_F \cdot (Bw, B)R_{J_1, J_2}/R_{J_1, J_2}$$

and for any $w \in \mathcal{W}(J_1, J_2)$, $G_F \cdot (Bw, B)R_{J_1, J_2}/R_{J_1, J_2}$ is a single G_F -orbit.

Notice that if $G = GL_n$, then each $G \times G$ -orbit on \mathfrak{gl}_n is of the form $G \times G/R_{J_1, J_2}$, where $J_1 = \{1, 2, \dots, i-1\}$ and $J_2 = \{i+1, i+2, \dots, n-1\}$ for some i . In particular, there are finitely many G_F -orbits on \mathfrak{gl}_n . These orbits classify the isomorphism classes of restricted abelian Lie algebras of dimension n . Thus we obtain the following unpublished result of Lin [Li].

THEOREM 4.1. *There are only finitely many isomorphism classes of restricted Lie algebra structure on an abelian Lie algebra of dimension n .*

5. Partial conjugation: affine type

5.1. Let $L = \mathbb{k}((\epsilon))$ be the formal Laurent series and $\mathfrak{o} = \mathbb{k}[[\epsilon]]$ be the ring of formal power series. Let $K = G(\mathfrak{o})$ be a maximal bounded subgroup of the loop group $G(L)$. Let I be the inverse image of B^- under the projection map $K \mapsto G(\mathbb{k})$ sending ϵ to 0 and K_1 be the kernel of this projection map. Let $\tilde{W} = N(T(L))/(T(L) \cap I)$ be the extended affine Weyl group of $G(L)$. It is known that $\tilde{W} = W \ltimes X_*(T) = \{w\epsilon^\chi; w \in W, \chi \in X_*(T)\}$, where $X_*(T)$ is the coweight lattice.

We only discuss in the section the partial conjugation of K on $G(L)$. The partial conjugation of arbitrary parahoric subgroup on $G(L)$ can be found in [L2].

Let \tilde{W}^S be the set of minimal coset representatives in \tilde{W}/W . Define $K_w = K \cdot IwI$ for $w \in \tilde{W}^S$ and call it a K -stable piece of $G(L)$. Here \cdot means conjugation action. Similar to our discussion in §3, we have that

$$(1) \quad G(L) = \sqcup_{w \in \tilde{W}^S} K_w. \text{ See [L2].}$$

(2) Let $R = \{(gu, gu'); g \in G(\mathbb{k}), u, u' \in K_1\}$. Then there is a natural bijection between the R -orbits on K_w and the $\text{Ad}(w)$ -twisted conjugacy class of $L_{I(S, w, id)}(\mathbb{k})$. See [L2].

$$(3) \quad \text{For any } w \in \tilde{W}^S, \overline{K_w} = \sqcup_{w' \in \tilde{W}^S, w' \leq_{S, id} w} K_{w'}. \text{ See [H5].}$$

The Frobenius-twisted case is studied by Viehmann in [V2].

5.2. In this subsection, we assume that G is an adjoint group. The specialization $\epsilon \mapsto 0$ defines a map s from the loop group $G(L)$ to the wonderful compactification X of $G(\mathbb{k})$. This specialization map was introduced by Springer in [Sp2, 2.1]. It is easy to see that $s(K) = G$ and $s(I) = B^-$.

Notice that any element in \tilde{W}^S is of the form $x\epsilon^{-\lambda}$ for some dominant coweight λ and $x \in W^{I(\lambda)}$, where $I(\lambda) = \{i \in S; \langle \lambda, \alpha_i \rangle = 0\}$. Now we have the following correspondence between the K -stable piece in $G(L)$ and the G -stable piece in the wonderful compactification X of $G(\mathbb{k})$. See [H5].

- (1) For any $\lambda \in Y^+$ and $x \in W^{I(\lambda)}$, $s(K_{x\epsilon^{-\lambda}}) = X_{I(-w_0\lambda), w_0xw_0}$.
- (2) For any $J \subset S$ and $x \in W^J$, $s^{-1}(X_{J,x}) = \sqcup_{\lambda \in Y^+, I(\lambda) = -w_0J} K_{w_0xw_0\epsilon^{-\lambda}}$.

This correspondence is a key ingredient in [H5] in the study of the relation between the closure of unipotent variety of $G(\mathbb{k})$ in X and the affine Deligne-Lusztig varieties in the affine flag of the loop group $G(L)$.

6. Conjugation action in loop groups

6.1. It is a challenging problem to study the case where J is not of finite type in the setting of Section 2.

In this section, we only consider the special case where $J = S$ is of affine type. More precisely, we consider a “twisted” conjugation action of $G(L)$ on itself as $g \cdot_\sigma h = \sigma(g)hg^{-1}$ for $g, h \in G(L)$. Here σ is a bijective group homomorphism on $G(L)$ of one of the following type:

- (1) For any nonzero element $a \in \mathbb{k}$, define $\sigma_a(p(\epsilon)) = p(a \cdot \epsilon)$ for any formal Laurent power series $p(\epsilon)$. We extend σ_a to a group homomorphism on $G(L)$, which we still denote by σ_a .
- (2) If \mathbb{k} is of positive characteristic and $F : \mathbb{k} \rightarrow \mathbb{k}$ is a Frobenius morphism. Then set $F(\sum a_n \epsilon^n) = \sum F(a_n) \epsilon^n$. We extend F to a group homomorphism on $G(L)$, which we denote by σ_F .

The σ_a -conjugacy classes were studied by Baranovsky and Ginzburg in [BG]. The σ_F -conjugacy classes were studied by Kottwitz in [Ko].

For simplicity, we focus on the case where $G = GL_n$. In this case, $\tilde{W} = S_n \ltimes \mathbb{Z}^n$. The discussion for other groups can be found in [H6] and [GH].

6.2. Similar to our discussion in the previous sections, we will use some special properties of the affine Weyl group to understand the twisted conjugation action in loop groups.

We call an element $w \in \tilde{W}$ a *good element* if $\ell(w^n) = n\ell(w)$ for all $n \in \mathbb{N}$ and we write \tilde{W}_{good} for the set of all good elements in \tilde{W} .

For any $J \subset \tilde{S}$, set $\mathcal{W}_J = \{wx; w \in \tilde{W}^J, x \in W_{I(J,w,id)}\}$. By Theorem 1.4, any W_J -conjugacy class in \tilde{W} contains an element in \mathcal{W}_J that is a minimal length element in that conjugacy class. We call a conjugacy class \mathcal{O} of \tilde{W} a *distinguished conjugacy class* if $\mathcal{O}_{\min} \cap \mathcal{W}_J \subset \tilde{W}^J$ for all $J \subsetneq \tilde{S}$.

The following result was proved in [H6].

THEOREM 6.1. *Let \mathcal{O} be a conjugacy class in \tilde{W} . Then*

- (1) *For any $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\min}$ such that $w \rightarrow w'$.*
- (2) *\mathcal{O} is distinguished if and only if \mathcal{O} contains a good element. In this case, $\mathcal{O}_{\min} \subset \tilde{W}_{good}$.*
- (3) *If \mathcal{O} is distinguished, then for any $w, w' \in \mathcal{O}_{\min}$, $w \approx w'$.*

6.3. For any distinguished conjugacy class \mathcal{O} of \tilde{W} , we choose a minimal length element $w_{\mathcal{O}}$ in \mathcal{O} . By [H6, 8.3], we introduce a partial order on the set of distinguished conjugacy classes as follows. We write $\mathcal{O} \leq \mathcal{O}'$ if there exists $w \in \mathcal{O}_{\min}$ such that $w \leq w_{\mathcal{O}'}$. It is showed that this definition is independent of the choice of the representative $w_{\mathcal{O}'}$.

In general, for any $x \in \tilde{W}$, we write $\mathcal{O} \leq x$ if there exists $w \in \mathcal{O}_{\min}$ such that $w \leq x$.

Based on these special properties of \tilde{W} , we have the following result. See [H6, Section 11].

THEOREM 6.2. *For $\sigma = \sigma_a$ or σ_F , we have that*

- (1) $G(L) = \sqcup_{\mathcal{O} \text{ distinguished}} \overline{G(L)} \cdot_{\sigma} I w_{\mathcal{O}} I$.
- (2) For any $x \in \tilde{W}$, $\overline{G(L)} \cdot_{\sigma} I x I = \sqcup_{\mathcal{O} \text{ distinguished}, \mathcal{O} \leq x} G(L) \cdot_{\sigma} I w_{\mathcal{O}} I$.

REMARK. If $\sigma = \sigma_F$, then for any good element w , $G(L) \cdot_{\sigma} I w I$ is a single σ_F -conjugacy class of $G(L)$. In this case, part (1) of the theorem is a reformation of Kottwitz's classification of σ_F -conjugacy classes of $G(L)$. The closure relation of σ_F -conjugacy classes was also obtained by Viehmann in [V2].

6.4. If $\sigma = \sigma_F$, then each σ -conjugacy class of $G(L)$ contains a representative b , here b is a block diagonal matrix and each block is of the form

$$\begin{pmatrix} 0 & \epsilon^{k_i+1} I_{k'_i} \\ \epsilon^{k_i} I_{n_i-k'_i} & 0 \end{pmatrix}.$$

The image \underline{b} of b in \tilde{W} is in general, not a good element. However, the S_n -conjugacy class of \underline{b} contains a unique element in \tilde{W}^S . That element is a good element associated to the σ -conjugacy class of b .

6.5. In the rest of this article, we discuss some recent progress on the affine Deligne-Lusztig varieties.

By definition, the affine Deligne-Lusztig varieties associated to an element w in the extended affine Weyl group \tilde{W} and a base point $b \in G(L)$ is

$$X_w(b) = \{gI \in G(L)/I; g^{-1}b\sigma_F(g) \in IxI\}.$$

They play an important role in the study of Shimura varieties. More details can be found in the survey papers of Görtz [Go], Haines [Ha] and Rapoport [Ra].

6.6. Define

- $\eta_1 : \tilde{W} = W \ltimes X_*(T) \rightarrow W$, projection map.
- $\eta_2 : \tilde{W} \rightarrow W$ such that $\eta_2(x)^{-1}x$ lies in the dominant Weyl chamber.
- $\eta(x) = \eta_2(x)^{-1}\eta_1(x)\eta_2(x)$.

Then we have the following result describing the dimension of $X_w(b)$.

THEOREM 6.3. *Let $b \in G(L)$ be a basic element, i.e., a length 0 element in \tilde{W} . Let w be an element in the lowest two-sided cell of \tilde{W} . Then $X_w(b) \neq \emptyset$ if and only if b and w are in the same component of $G(L)$ and $\eta(w)$ is not in any proper standard parabolic subgroup of W . If moreover, the translation part of w is regular, then*

$$\dim X_w(b) = \frac{1}{2}(\ell(w) + \ell(\eta(w)) - \text{def}(b)),$$

here $\text{def}(b) = \text{rank}(G) - \text{rank}(J_b)$, where J_b is the σ_F -centralizer of b .

The “only if” part of the theorem was proved by Görtz, Haines, Kottwitz and Reuman in [GHKR2] and the remaining part was proved in [GH]. In loc. cit., the upper bound $\dim X_w(b) \leq \frac{1}{2}(\ell(w) + \ell(\eta(w)) - \text{def}(b))$ was deduced from the dimension formula in affine Grassmannian [GHKR1] and [V1]. Theorem 6.1 is a key ingredient to establish the lower bound $\dim X_w(b) \geq \frac{1}{2}(\ell(w) + \ell(\eta(w)) - \text{def}(b))$.

6.7. Notice that $X_w(b) = \varinjlim X_i$ for some closed subschemes $X_1 \subset X_2 \subset \cdots \subset X_w(b)$ of finite type. Thus we may define the Borel-Moore homology of $X_w(b)$ as

$$H_j^{BM}(X_w(b), \bar{\mathbb{Q}}_l) = \varinjlim H_c^j(X_i, \bar{\mathbb{Q}}_l)^*,$$

here l is a prime number not equal to the characteristic of \mathbb{k} . The action of the σ_F -centralizer J_b of b on $X_w(b)$ induces a smooth representation of J_b on $H_j^{BM}(X_w(b), \bar{\mathbb{Q}}_l)$. The following property was obtained in [H6]. The proof is again based on Theorem 6.1.

THEOREM 6.4. *Let μ a dominant regular coweight of $G = GL_n$. Then $J_{\epsilon^\mu} = T(\mathbb{F}_q((\epsilon)))$ and $T(\mathbb{F}_q[[\epsilon]])$ acts trivially on $H_j^{BM}(X_w(\epsilon^\mu), \bar{\mathbb{Q}}_l)$ for all $w \in \tilde{W}$ and $j \in \mathbb{Z}$.*

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DEPARTMENT OF MATHEMATICS, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, KOWLOON, HONG KONG
E-mail address: macyfang@ust.hk

DEPARTMENT OF MATHEMATICS, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, KOWLOON, HONG KONG
E-mail address: maxhhe@ust.hk